THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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OBITUARY.

R. W. H. T. HUDSON.

It is with the deepest sorrow that we record the tragic death of Ronald Hudson, at the early age of 28, while spending a holiday with a friend in Wales. He and his friend started from the Penygwryd Hotel early on Tuesday morning, September 20, and after climbing one of the Glydyrs descended to the foot of the Devil's Kitchen by Lake Idwal, which they reached about one o'clock. Ronald Hudson was an enthusiastic rock climber, and would not be dissuaded from trying part of the ascent, although his companion was not sufficiently expert to go with him. A short half hour later he apparently dislodged a large rock, fell a considerable distance, and in all probability was instantly killed. When a search party with much difficulty found him some time after, he was dead. On the following Saturday he was laid to rest in Wandsworth Cemetery, by the side of his mother, whom he had lost when he was six years old. The heartfelt sympathy of his large circle of friends goes out to the sorrowing family, to his father, aunt and sisters, who have followed along the same path of knowledge as their brother with no less determination and success.

It is impossible not to associate this terribly sad accident with that which befell his intimate friend A. P. Thompson little more than a year ago. They were educated in the same school, but were never in the same class; and at Cambridge they were drawn together not so much by their common devotion to mathematics as by their love of music. On the evening of the day that proved fatal to Thompson they had arranged to go over some new music together. Hudson often repeated that he owed much to his friendship with Thompson, and there is no doubt that each exerted a profound influence on the character of the other.

Ronald William Henry Turnbull Hudson was named after his father, Professor W. H. H. Hudson, and his grandfather, Robert Turnbull; but his

first name was that by which he was best known and will always be remembered. He was born at Cambridge on July 16th, 1876, was the eldest of four children, and the only son. At the age of 12 he went to St. Paul's School, and before he was 15 was transferred from the Classical to the Mathematical 8th. In 1895 he entered St. John's College, Cambridge, was Senior Wrangler in 1898, and bracketed Smith's Prizeman with J. F. Cameron in 1900. The same year he was elected Fellow of his College, and shortly after declined the offer of an important post, the acceptance of which would have meant the abandonment of his favourite studies. In 1902 he became Lecturer at University College, Liverpool, and Secretary of the Mathematical Section of the British Association, and in 1903 was awarded the D.Sc. degree at London University.

His mathematical writings form a remarkable record for four short years of work after completing the University course. The following list, with the exception of the first paper, was recently made out by himself.

Mineralogical Magazine, 1900, vol. 12, p. 58. "On the Determination of the Positions of Points and Planes after Rotation through a definite Angle about a known Axis"; afterwards translated in the *Zeitschrift für Krystallographie, etc.*, 1901, Bd. 34, Heft 4.

Messenger of Mathematics.

Vol. 29, p. 191. Note on Reciprocation.

" 31, p. 151. A New Method in Line Geometry.

" 31, p. 159. Note on the conditions of Equilibrium of a Flexible Membrane under Hydrostatic Pressure.

" 32, p. 31. Dual Line Coordinates in Absolute Space.

" 32, p. 51. Matrix Notation in the Theory of Screws.

" 33, p. 50. The Surface of Floatation.

Proceedings of the London Mathematical Society.

Vol. 33, p. 269. On Discriminants and Envelopes of Surfaces.

" 33, p. 380. A Geometrical Theory of Differential Equations of the First and Second Orders.

" 34, p. 154. The Puiseux Diagram and Differential Equations.

The Mathematical Gazette.

Vol. 2, p. 279. An Elementary Introduction to the Infinitesimal Geometry of Surfaces.

" 2, p. 354. The Use of Tangential Coordinates.

" 3, p. 56. Univocal Curves and Algebraic Curves on a Quadric Surface.

Quarterly Journal of Pure and Applied Mathematics.

Vol. 34, p. 98. The Discriminant of a Family of Curves or Surfaces (jointly with T. J. I'a Bromwich).

Bulletin of the American Mathematical Society.

Vol. 9, p. 308. The Analytic Theory of Displacements.

In addition to the above he gave abstracts of some of his writings at meetings of the British Association, and wrote several reviews of important

works for *Nature* (vols. 67, 68, 69) and the *Mathematical Gazette*. He had finished a treatise on "Kummer's Quartic Surface," which will be published by the Cambridge University Press. He had just commenced a book on Analytical Geometry, and had been requested to write another on Elementary Pure Geometry, which it was hoped he would undertake later.

Ronald Hudson's was a singularly strong, noble and lovable personality. The writer of this note was attached to him by long and intimate friendship, and experienced many an act of thoughtful kindness at his hands. It is impossible to write of him anything but a bare record while the thought of his death is so near and overwhelming. We can scarcely yet realise that one who was so clearly destined to win fame if he had lived has already gone from us. F. S. M.

THE TEACHING OF MATHEMATICS AND PHYSICS.*

THE Association of Public School Science Masters fixed their annual meeting for Jan. 16, and accepted a proposal by Mr. R. H. Thwaites to read a paper for discussion at that meeting on the subject of the possible fusion of the teaching of mathematics and science in public schools. I wrote to Mr. Pendlebury, pointing out the interest of the subject to us, and suggesting that our meetings ought not to clash. In reply, he requested me to open a similar discussion at our meeting to-day. I hope I have made it clear that the suggestion that the subject is ripe for consideration is originally due to Mr. Thwaites.

As some of us were present last week at the Science meeting; and as the discussion there has been fully reported, I have taken advantage of Mr. Pendlebury's alternative permission to continue the discussion rather than to attempt any detailed account of what then took place. I am very conscious that I am risking the commission of two grave errors—I may be preaching to the converted—and, even worse, I may appear to be trying to instruct those who know more about the matter in hand than I do myself. On these grounds I ask for your indulgence.

It seems to me that there are two main reasons why we who are as a body teachers of elementary mathematics should be glad to consider favourably any suggestion of the kind under discussion. In the first place, there are signs that the true function of examinations is beginning to be more clearly realised. Recent events have been very gratifying to those of us who hold that the examiner is an excellent partner, but a very bad master, for the teacher. But with the decay and disappearance of the mechanical examination system there will pass away a powerful inducement to work and a bond of sympathy between the teacher and the pupil, who will no longer find themselves in alliance against their common enemy, the examiner. Failing the unhealthy stimulus of examination pressure, we shall have to do our utmost to develop and utilise the natural stimulus of keen interest in the subject.

The second reason to which I refer as supporting the introduction of experimental work is this. Professor Pascal has observed that the movement towards increased rigour of demonstration has led in almost all branches of mathematics to minute and critical discussions, in the course of

* A paper opening a discussion at the annual meeting of the Mathematical Association Jan. 23, 1904.

which propositions, formerly treated as universal and almost axiomatic, have been shown to depend upon subtle and complex conditions. This has unquestionably created a good deal of embarrassment for teachers of elementary mathematics. It is impossible to impose rigour on the beginner, and most undesirable to attempt it. No less an authority than Professor Klein can be cited in support of this view, which, in fact, constitutes the principle underlying the recent reforms in the teaching of geometry.

Even at the Ecole Polytechnique, where the students can hardly be called beginners, the attempt which was made to treat the Calculus from the outset on lines of strict rigour, had, I am told, to be abandoned. No doubt in commencing the Calculus one may now and again draw attention to some mathematical curiosity, such as that which Professor Pascal adopts as his first example:

$$\underset{t=0}{L^t}\ \underset{n=\infty}{L^t} \frac{\sin^2(x\,.\,n'\,.\,\pi)}{t^2+\sin^2(x\,.\,n'\,.\,\pi)},$$

which is zero when x is commensurable and unity when x is incommensurable. But if we are to put aside refinements which a beginner cannot appreciate, we shall find it very difficult to avoid enunciating theorems as generally true without regard to the qualifications under which alone they are true. In our difficulty we may turn for help to history. The evolution of the individual is a recapitulation of the evolution of the race. Now a science begins with facts and special methods, and ends with critical exactness and with generalised methods. As teachers of pre-university work, it is our main business to familiarise our pupils with facts and the mode of dealing with facts.

To my mind, the due discrimination on broad lines between the work proper to a school and the work proper to a university is a problem which calls for our most serious attention, and the solution of which will dispose of many of our difficulties. As teachers of school work, we must frequently be content with the certainty that a method is available in a particular case. Thus I should advocate the use of the method of differentiation in obtaining particular expansions, whose convergency may be examined in each individual case rather than purporting to furnish any general method of expanding functions. We shall have to lay increased stress on intuition. With that tendency to fetishism which most of us inherit, we have been sometimes apt to unconsciously attribute a sort of objective value to certain "pieces of book-work." To reproduce Duchayla's proof was to perform a meritorious action.

Under the influence of Mach we have come to doubt the objective value of this or any similar performance, and to consider rather whether the "proof" has really helped the student to realise more vividly, distinctly, and precisely that the proposition is true. Has it aided him to form a useful 'concept'?

It must, of course, be acknowledged that the truth and beauty of a demonstration may flash upon the mind after, but not until after, long study. But these matters are not for the beginner, whose mathematical courage must not be damped by premature encounters with difficulties which he cannot surmount. Further, one of the idlest fears that can beset a teacher is the notion that he can possibly make his demonstrations too easy or his explanations too clear. Now, it must, I think, be admitted that in many cases models and experiments may be employed with advantage in aiding a pupil to grasp a demonstration and in developing mathematical insight. Very often, as soon as a clear mind picture of the problem has been formed, its solution is obvious. The faculty of visualising a somewhat complicated object without the aid of models can be greatly developed by the judicious use of models, and it shares with mental arithmetic the distinction of being a most valuable mental power, and of being almost entirely uncared for in our educational system. Dr. Todhunter is said to

have observed that no one who required a model or experiment to enable him to grasp a geometrical or mechanical theorem could ever be a mathematician. I believe that this is perfectly true. But how many of us who are lovers of mathematics would venture to call ourselves mathematicians? If one is not capable of being a mathematician in the high sense of the word, is one therefore to be prohibited from getting at the truth of things as best one can? I may perhaps give a personal reminiscence, which compromises no one but myself. When I went in for the mathematical tripos, I had no real conception of the action of a convex lens. I could have doubtless worked out various questions. But if you had said, "A candle is placed in front of a convex lens, and a real inverted image formed on a screen. The eye is then placed at the point where the image was formed. What will the eye see?" I could not have told you.

But, I may be told, you are asking that proposals for change should be welcomed without telling us what the proposals are. I am not inclined to go very far in the way of specific suggestion at present. The mode of teaching chemistry and physics, and, in particular, the proper extent and aim of a school course in physics, is still far from being settled. I think teachers of mathematics may play a very useful part in the discussion, particularly in encouraging simplicity of method. Apart, however, from the general statement that teaching of geometry, trigonometry, and, in particular, mechanics, has, I submit, much to gain by becoming more concrete, I will venture on one specific suggestion, namely, that for young boys the study of Heat should be replaced by the study of Optics, and that the experimental work in Optics should be under the control of the mathematical staff. The laws of reflection and refraction furnish us with admirable and interesting exercises in practical geometry; and this work should give definiteness of ideas which will help the subsequent study of aberration, if that is reached at the university stage. Lenses are cheap; rough, but working, models of a sextant, a low-power telescope, an optical level or square and the like, are easily made. I produce a rough model of a sextant, not as being worthy of exhibition—for, if put forward with that intent, it would justly arouse ridicule—but in support of my contention that, with the roughest materials and the simplest tools, a boy of 12 could construct in half an hour a working model capable of measuring angles within a degree, and that a boy of more than 12 could learn a good deal from it. The problem how to make tenth of an inch section paper furnish a scale of degrees gives a little exercise in trigonometry. I attach importance to making rough models with one's own hands. Centering error, index error, error due to the mirrors not being at right angles to the plane of the arc, are very real when, for example, the latter error throws the image right off the second mirror.

I learn from Mr. Fletcher that he has put these views as to the teaching of optics into practice at Liverpool, for some time past, with complete success. If the treaty of alliance Mr. Thwaites proposes is entered into, the teaching of physics will become more mathematical; and here, at any rate, we believe that this will be well. On the other hand, teachers of mathematics will have to lay more stress on the art of stating problems, which is at least as important as the art of solving them. It has been said that hardly any person would fail to solve a strategical problem correctly if it were stated to him correctly, and divested of all irrelevant details. But the task of the commander is, as I suppose, out of the maze of facts, rumours, reports, surmises, and possibilities, to divine what the real problem is; and this is the task of the investigator in most subjects. Mrs. Boole has spoken in a delightful book of the importance of drawing attention, during the early stages of education, to the question whether a particular fact is or is not relevant for a particular purpose.

I should like to give a couple of concrete instances, one on each side, of the kind of misapprehension which a judicious admixture of theoretical and practical work would tend to prevent. The first I give simply as a legend, not to say a libel. When buffers were first proposed to take up the recoil of a gun on firing they were objected to; and it is said that their adoption was delayed on the ground that the explosion of the charge communicated a certain definite momentum to the gun, and that the whole of this momentum must be equally communicated, so to speak, to the ground, irrespective of any details of internal construction of the gun carriage. Hence, buffers are useless. Here was a conclusion based upon premises which were true enough. We, however, see at once that the *rate* of destruction of momentum is the decisive feature of the problem, and this is just what the buffer alters. If the objector would have jumped from a height of thirty feet, first into a net and then on to a stone pavement, losing the same amount of momentum in each case, he might, either in this world or the next, have reconsidered his argument.

On the other hand, in remonstrating with a student for producing some entirely new views on specific gravity, I once said, "But you must have often done this experimentally at school." The answer was: "Yes, sir: but we always had the formula given us, and only had to think about the weights." I do not much, if at all, object to his having the formula given him, but I do object to his only thinking about the weights. We must not forget that, while experiment is good, the discoverer thinks *before* he experiments as well as after. Here, again, I think teachers of mathematics may exert a beneficial influence. Some people fear that that the fusion of physics and mathematics in public school teaching may be like the fusion of the tiger and the lamb. But I hope I have shown that the high contracting parties will meet on a footing of equality, and should both benefit equally. An apprehension is sometimes expressed as to the cost of laboratories and appliances for teaching physics. I am happy to think that many people whose opinion on a question of science is of real weight, agree with me in thinking that we have had far too much in the past of the "instrument on polished mahogany base, with all brass parts richly lacquered," for demonstrating that a weight hung up by a string will oscillate to and fro if disturbed from its position of equilibrium. Home-made apparatus—cardboard, seccotine and scissors, sixpenny lenses in cardboard tubes, paper scales, levers and pulleys made in the school workshops—are what we want. The fact is often lost sight of how rough, comparatively speaking, were some of the methods and results of the great pioneers.

Mankind may be divided roughly into two classes: those who understand their job and those who do not, generally because they do not know what understanding a subject means. Our first object as teachers is to teach people what understanding means, in order that in after life they may try to belong to the former class, be their job what it may.

I have attempted to put before you the arguments which have convinced me that a judicious admixture of mathematics and physics will aid us more powerfully than stronger separate doses of each to effect our object.

C. S. JACKSON.

ON THE AXIOMS AND POSTULATES EMPLOYED IN THE ELEMENTARY PLANE CONSTRUCTIONS.

BY the elementary plane constructions we mean the bisection of angles and segments of lines and the drawing of perpendiculars. The constructions given below are extremely simple, and it is very possible that some or all of them have been given before; but they seem to be of sufficient interest to

be reproduced, for the reason that, if they are not new, they are not generally known.

The axioms employed are those tacitly assumed or explicitly stated in Euclid in relation to straight lines and angles, the axiom of parallels and the axiom that "two straight lines cannot enclose a space" being excepted. This will probably be accepted as a sufficient explanation, since we are not at present concerned with any abstruse analysis of axioms.

For the axiom of superposition we may substitute the axiom that the plane is uniform, possessing identical properties throughout, and that its uniformness about a point is independent of phase or sense of rotation. From this axiom many propositions follow at once, such as the equality of vertically opposite angles, the equality of the base angles of an isosceles triangle with its converse, and the identical equality of triangles having certain parts equal. Other propositions can be proved, such as the identical equality of triangles with all three sides given, or that of right-angled triangles having given the hypotenuse and one side, or a side and the opposite angle. It also follows from the axiom that all or none of the straight lines of the plane are closed. If they are not closed two straight lines through a point A do not cut again. If they are closed two straight lines through A may or may not cut again in another point B. If they do cut again in B, then every straight line through A passes through B; the lines AB are all of equal length; and they are all perpendicular to and bisected by one and the same straight line.

The postulates employed are that a straight edge may be used for drawing the straight line through any two given points, and for prolonging a given segment of a line to any desired extent; and that the dividers may be used for transferring segments, that is, for marking off from any point of a straight line a length equal to that of a given segment.

The postulate that a circle may be described with any centre and any radius is not employed, so that no assumption is made about the intersections of circles.

The proofs of (2), (3), (4), (6) are evident, and are consequently omitted.

(1) *To bisect a given angle.* (Fig. 1.)

Let A be the vertex. Choose two points B, C in one arm, and from the other arm cut off $AD=AB$, and $AE=AC$. Join BE, CD, cutting in X. Join AX; then AX bisects the angle A.

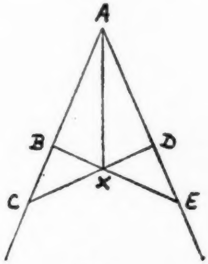


Fig. 1.

If straight lines have at most one point of intersection this construction is unique. If they have two points of intersection, there will be two points X, and the two lines AX will bisect the angle A internally and externally. Also, in this case, in order to prevent the possibility of the construction

failing, the lengths AB and AC should both be chosen less than half that of a complete line. The proof is by showing that $XB=XD$, since

$$X\hat{B}D=E\hat{B}A-D\hat{B}A=C\hat{D}A-B\hat{D}A=X\hat{D}B.$$

(2) *To draw a perpendicular to a given straight line.** (Fig. 2.)

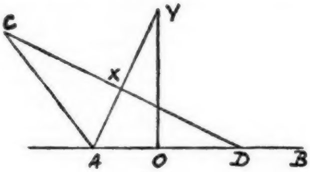


Fig. 2.

Let AB be the given line. Draw another line AC, and cut off AD from AB equal to AC. Bisect the angle CAD by a line meeting CD in X. Cut off AO from AB equal to AX, and AY from AX equal to AD. Join YO; then YO is perpendicular to the given line AB.

(3) *To draw a perpendicular to a given line from a given point on it.* (Fig. 3.)

Let AB be the line, and C the point. Draw a perpendicular YOZ to AB, by (2), meeting it in O, and make OY equal to OZ. Join ZC, YC; and produce YC to X. Bisect the angle ZCX, by (1); then the bisector is the required perpendicular.

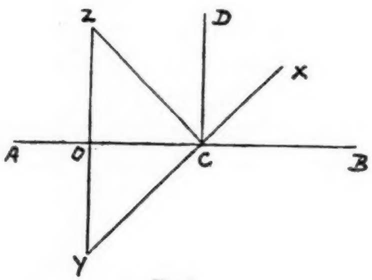


Fig. 3.

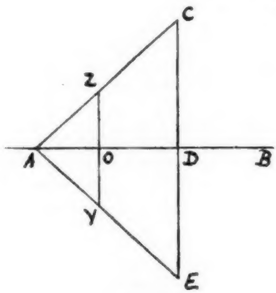


Fig. 4.

If straight lines are closed, a simpler construction can be given, viz.: draw two perpendiculars to AB, by (2), cutting in P, and join CP. This applies whether C is on or off the line.

(4) *To draw a perpendicular to a given line from a given point without it.* (Fig. 4.)

Let AB be the line, and C the point. Join CA. If CA, and also CB, are perpendicular to AB, then every line through C is perpendicular to AB. If CA is not perpendicular to AB, draw a perpendicular to AB from a point O on it, by (3), so as to cut AC in a point Z. This can always be done by choosing O sufficiently near to A. Produce ZO to Y,

* This construction, and a similar one, were supplied to the writer by two of his pupils, H. J. Higgs and R. H. J. Sasse.

making OY equal to OZ. Join AY, and produce it to E, making AE equal to AC. Join CE; then CE is the required perpendicular.

One perpendicular can always be drawn from C to AB, and not more than one, unless every line through C is perpendicular to AB.

(5) *To bisect a given segment.* (Fig. 5.)

Let AB be the segment. Draw AC, BD perpendicular to AB *on opposite sides*, and make AC equal to BD. Then CD cuts the segment AB at its middle point O.

If straight lines are closed, C must not be a point such that every line through C is perpendicular to the line AB, and C must not be at the second

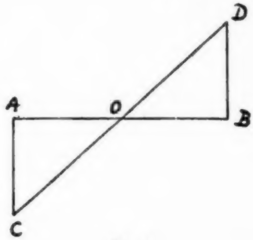


Fig. 5.

point of intersection of the lines AC, AB if there is one. The proof is by first showing that $OC=OD$; and then showing that if OA is not equal to OB a perpendicular can be drawn from C to AB different from CA, which is impossible. An open segment AB has always two *opposite sides*, although a closed straight line may not have two sides.

(6) *To draw a line making with a given line an angle equal to a given angle.* (Fig. 6.)

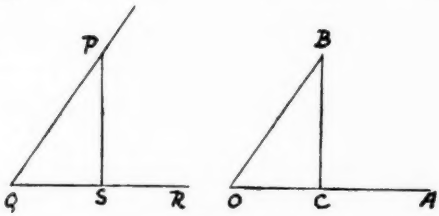


Fig. 6.

Let OA be the given line, and PQR the given angle. From a point P on QP draw PS perpendicular to QR. From OA cut off OC equal to QS. Draw a perpendicular CB to OA, and make CB equal to SP. Join OB; then AOB is the required angle.

F. S. Macaulay.

NOTE ON THE ADDITION FORMULAE OF TRIGONOMETRY.

Proofs of the addition formulae for angles of any magnitudes are sometimes rendered invalid by the way in which arguments are founded on a diagram which can represent only one of the many possible cases. When the two

angles under consideration are represented in the usual manner as having a common vertex, the only angles in the diagram whose values it is safe to represent by formulae, if complete generality is to be preserved, are angles having this same vertex. The introduction of other angles may be avoided by a restatement of the definitions of sine and cosine.

At the outset some remarks about the specification of an angle suggest themselves.

(1) When we speak of the angle between two *lines* we mean the angle between two *directions*; it is therefore desirable, even in the most elementary teaching, to postulate that the trigonometrical line, that is a line regarded as an arm of an angle, shall always mean a directed line, a line with an arrow-head.

(2) The merely arithmetic idea of an angle is sufficient in elementary Trigonometry, and it is unwise to suggest to the beginner that an angle described in the clockwise sense must be regarded as negative; the table-book tells us the value of the sine of an angle without asking in which sense the angle is supposed to be described, and the learner ought to be led to take the same point of view. In elementry Trigonometry and in practical work the measure of an angle is an arithmetical quantity, subject to arithmetic addition and subtraction; the idea of algebraic addition, with the implied idea of a negative angle, is confusing to the beginner and may with advantage be postponed.

(3) Two directed lines are the arms of an infinite series of angles. When we speak of the angle between two lines we mean one of these infinitely numerous angles, and we must know which. The manner of obtaining this knowledge does not concern us at present, but it may be remarked here that in the most frequent application, namely to the triangle, our choice is conventionally directed to the interior angles. The essential thing is that we know what angle is to be discussed before we discuss it; the method of indicating an angle in a diagram by means of an arc of a small circle is, of course, most valuable.

Definitions.

Either arm of an angle may be called the *first* arm, and the angle may be regarded as described by a line starting from the *first* arm and revolving *through the angle* to the *second* arm. The sense in which the revolution takes place depends on which arm we select as the first arm. In order to define the sine and the cosine a choice must be made of the order in which the sides are to be taken, but the choice is quite arbitrary.

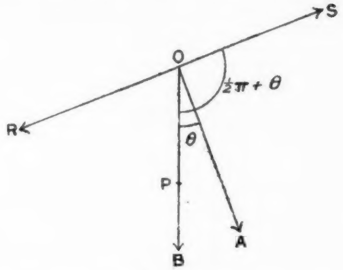
Let OA, OB be the arms of an angle θ, and let OA be chosen as the first arm. Of the two directed lines through O at right angles to OA let OR be that which is such that the right angle AOR and the angle θ would be described in the same sense by a revolving line starting from OA; the directed line OR may be called the *right-angle arm* corresponding to the angle θ. But it will perhaps be more convenient to call OA the *cosine arm*, OB the *second arm*, and OR the *sine arm* of the angle θ. Of course if OB were taken as cosine arm we should get a different sine arm.

Now measure a unit distance OP from O along the second arm OB of the angle, in the direction of the arrow belonging to OB. The *cosine* of θ is defined to be the projection of OP on the cosine arm OA; the *sine* of θ is defined to be the projection of OP on the sine arm OR; projections are to be estimated algebraically, with reference to the arrows on OA and OR.

These definitions have the advantage that they make no mention of 'right,' 'left,' 'up,' 'down,' 'clockwise' or 'counterclockwise.' It follows from them that if OQ be any length, positive or negative, measured along the second arm of an angle θ, its projections on the cosine arm and sine arm respectively are algebraically equal to $OQ \cos \theta$ and $OQ \sin \theta$.

To find the sine and cosine of $\frac{1}{2}\pi+\theta$.

Let AOB be the angle θ, and let OA be taken as cosine arm. Let OR be the corresponding sine arm, and let RO be produced to S. Then the angle



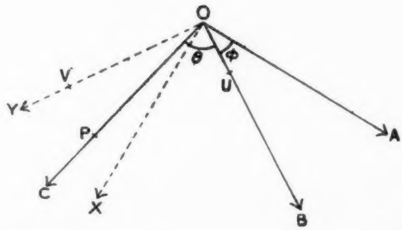
SOB is $\frac{1}{2}\pi+\theta$, and we may take OS for its cosine arm. Thus θ and $\frac{1}{2}\pi+\theta$ have the same second arm OB. Let P be in OB at unit distance from O.

Then OA is the sine arm of $\frac{1}{2}\pi+\theta$, and is the cosine arm of θ. Hence $\cos\theta$ and $\sin(\frac{1}{2}\pi+\theta)$ are both equal to the projection of OP on OA, and so are equal to one another.

Again OR is the sine arm of θ, and OS is the cosine arm of $\frac{1}{2}\pi+\theta$, and these are in opposite directions. The projection of OP on OS is *minus* the projection of OP on OR: hence $\cos(\frac{1}{2}\pi+\theta)=-\sin\theta$.

The Addition Formulae.

Let a straight line start from coincidence with OA and revolve about O always in the same sense. Let OB be its position when it has described an



angle ϕ, OC its position when it has described $\phi+\theta$, OX its position when it has described $\frac{1}{2}\pi$, and OY its position when it has described $\frac{1}{2}\pi+\phi$.

Then we may take OA, OB, OX for cosine arm, second arm, and sine arm respectively of the angle ϕ or AOB; we may take OB, OC, OY for cosine arm, second arm, and sine arm respectively of the angle θ or BOC; and we may take OA, OC, OX for cosine arm, second arm, and sine arm respectively of the angle $\phi+\theta$ or AOC.

Let P be at unit distance from O on OC, and let OU, OV be the projections of OP on OB and OY; of course UP and OV are parallel and in the same direction.

The cosine of $\phi+\theta$ is the projection of OP on OA; this is the same as the sum of the projections of OU and UP (or OV) on OA. But OU is $\cos\theta$, and

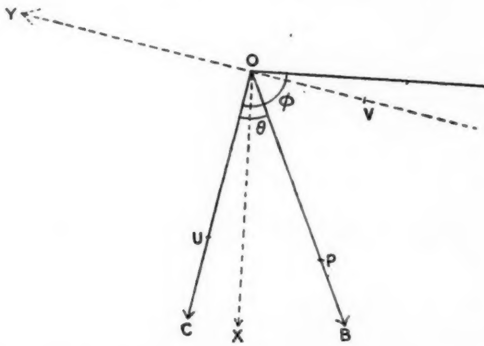
is estimated positively along the second arm OB of the angle ϕ whose cosine arm is OA; so the projection of OU on OA is $\cos\theta \cos\phi$. And OV is $\sin\theta$, and is estimated positively along the second arm OY of the angle $\frac{1}{2}\pi+\phi$ whose cosine arm is OA; so the projection of OV on OA is $\sin\theta \cos(\frac{1}{2}\pi+\phi)$.

Hence
$$\cos(\phi+\theta)=\cos\theta\cos\phi+\sin\theta\cos(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

The sine of $\phi+\theta$ is the sum of the projections of OU and OV on OX. Now OX is the sine arm of the angle ϕ along whose second arm lies OU, and also of the angle $\frac{1}{2}\pi+\phi$ along whose second arm lies OV. Hence
$$\sin(\phi+\theta)=OU\sin\phi+OV\sin(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\sin\phi+\sin\theta\cos\phi.$$

The Subtraction Formulae.

Let a straight line start from coincidence with OA and revolve about O, always in the same sense. Let OB be its position when it has described an



angle $\phi-\theta$, OC its position when it has described an angle ϕ, OX its position when it has described $\frac{1}{2}\pi$, and OY its position when it has described $\frac{1}{2}\pi+\phi$.

Then we may take OA, OB, OX for cosine arm, second arm, and sine arm of the angle $\phi-\theta$ or AOB; we may take OA, OC, OX for cosine arm, second arm, and sine arm of the angle ϕ or AOC; and we may take OC, OB, and the line OY *reversed* as cosine arm, second arm, and sine arm of the angle θ or BOC.

Let P be at unit distance from O along OP, and let OU, OV be the projections of OP on OC and OY respectively. Then OU is $\cos\theta$; but OV, being estimated positively along OY which is the reversed sine arm of θ, is $-\sin\theta$.

The cosine of $\phi-\theta$ is the projection of OP on OA; this is the same as the sum of the projections of OU and UP (or OV) on OA. But OU is measured along OC, the second arm of the angle ϕ whose cosine arm is OA; so the projection of OU on OA is $OU\cos\phi$, or $\cos\theta\cos\phi$. And OV is measured along OY, the second arm of the angle $\frac{1}{2}\pi+\phi$ whose cosine arm is OA; so the projection of OV on OA is $OV\cos(\frac{1}{2}\pi+\phi)$ or $-\sin\theta\cos(\frac{1}{2}\pi+\phi)$. Hence
$$\cos(\phi-\theta)=\cos\theta\cos\phi-\sin\theta\cos(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\cos\phi+\sin\theta\sin\phi.$$

The sine of $\phi-\theta$ is the sum of the projections of OU and OV on OX. Now OX is the sine arm of the angle ϕ along whose second arm lies OU, and also of the angle $\frac{1}{2}\pi+\phi$ along whose second arm lies OV. Hence

$$\begin{aligned}\sin(\phi-\theta)&=OU\sin\phi+OV\sin(\tfrac{1}{2}\pi+\phi)\\&=\cos\theta\sin\phi-\sin\theta\sin(\tfrac{1}{2}\pi+\phi)\\&=\cos\theta\sin\phi-\sin\theta\cos\phi.\end{aligned}$$

Negative Angles.

The transition from arithmetic subtraction to algebraic addition is effected by giving to the sine and cosine of a negative angle such meanings as shall make the subtraction formulae particular cases of the addition formulae. It is easily seen that the required convention is simply that when an angle is to be regarded as negative the direction of its sine arm must be reversed.

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A SUGGESTED REARRANGEMENT OF THE BOOK-WORK ON SOME ELEMENTARY SERIES.

IN all the English text-books on Analytical Trigonometry, so far as I know, the power-series for $\cos x$ is obtained as the limit, when n tends towards* infinity, of the finite sum for $\cos x$,

$$\cos^n\left(\frac{x}{n}\right)\left[1-\left(1-\frac{1}{n}\right)\frac{x^2}{2!}t^2+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{x^4}{4!}t^4-\dots\right],$$

where $t=\left(\tan\frac{x}{n}\right)\Big/\frac{x}{n}$ and the number of terms is equal to the integer next greater than $\frac{1}{4}n$. The power-series for $\sin x$ is obtained similarly; the two series and the method of finding them being both due to Euler.

Now it appears to me that this method is subject to certain drawbacks, when introduced in a *first* course on Analytical Trigonometry. For there are two ways of calculating the limit; the harder (but accurate) one as given in books such as Chrystal's *Algebra* or Hobson's *Trigonometry*; and the easier (inaccurate) method of the older text-books, which has unfortunately been reproduced in some new texts.†

The weakness in the inaccurate form of the proof is found in the assumption that the product

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{2r-1}{n}\right)$$

has the limiting value 1, for *all* values of r which do not exceed $\frac{1}{2}n$ (see for instance pp. 109, 110 of Lachlan and Fletcher's book just quoted). The falsity of this assumption is almost self-evident, when thus stated; but the assumption is, nevertheless, nearly always swallowed by the beginner without a qualm.‡

* It is perhaps worth while to enter a plea for the use of the phrase, *from the very beginning*, in dealing with limits. The phrases "the limit when n *is* ∞," "the limit when $n=\infty$" can do nothing but harm.

† Castle's *Manual of Practical Mathematics* (Macmillan, 1903); Lachlan and Fletcher's *Elements of Trigonometry* (Arnold, 1904). I refer to these as examples, merely; in many other respects both books strike me as good.

‡ I have found a little numerical calculation very useful in convincing the unbeliever that there *is* an assumption involved. Thus with n even, and $r=\frac{1}{2}n$, we find the product to be less than $1/10^3$ when $n=10$; less than $1/10^{10}$ for $n=30$; less than $1/10^{20}$ for $n=50$.

On the other hand the accurate evaluation of the limit is often found hard to grasp, the essential points of the proof being obscured to a certain extent by the details of the algebra. In my opinion the real difficulty arises from the fact that a *double* limit is introduced in the course of the proof; and that an appeal is made to what is really the principle of *uniform convergence* (in a somewhat disguised form). For what is done amounts to this:—The finite sum for $\cos x$ is divided into two parts, the first m terms $\sigma(m, x, n)$ say, and the remainder $\rho(m, x, n)$. It is then proved that if x lies within certain bounds, $\rho(m, x, n)$ can be made numerically less than any arbitrarily prescribed fraction ϵ, by choice of *m only however great n may be* (provided that $n > 2m$). In other words $\rho(m, x, n)$ is proved to converge to zero *uniformly* for *all* values of x, n, subject to the conditions just mentioned. Next, the infinite series $S(x)$ is divided into the first m terms $S(m, x)$ and the remainder $R(m, x)$; and it is proved that for the *same* value of m, $R(m, x)$ is also numerically less than ϵ. Finally, m having been now *fixed*, it is proved that N can be found so that when $n > N$, the difference between $\sigma(m, x, n)$ and $S(m, x)$ is also less than ϵ in numerical value. Hence $\cos x - S(x)$ is numerically less than 3ϵ, provided $n > N$, but $\cos x - S(x)$ is *independent* of n, and consequently $\cos x - S(x) = 0$.

Now this skeleton proof is, for practical purposes of teaching, further complicated by certain algebraic inequalities which must be used in finding the greatest possible values of ρ, R and $\sigma - S$. And, so far as my experience goes, I have found it sufficiently hard to explain the idea of uniform convergence, even when the simplest examples are used, from which all algebraic difficulties have been removed; and it would therefore seem unreasonable to expect a beginner to follow a proof which depends on uniform convergence and involves certain further difficulties of a purely algebraic character.

For the reasons just explained, I have been led to abandon the use of these limits in a first course; and I am inclined to think that other teachers may have felt the same difficulties. My solution of the problem is to give a short course on the elementary Differential Calculus before touching on the sine and cosine power-series. Having explained that when a (*continuous*) function of x has a positive differential coefficient, the function increases with x, I take the following cases:

$$\text{(i)}\quad \frac{d}{dx}(x - \sin x) = 1 - \cos x,$$

$$\text{(ii)}\quad \frac{d}{dx}[\cos x - (1 - \tfrac{1}{2}x^2)] = x - \sin x,$$

$$\text{(iii)}\quad \frac{d}{dx}[\sin x - (x - \tfrac{1}{6}x^3)] = \cos x - (1 - \tfrac{1}{2}x^2),$$

$$\text{(iv)}\quad \frac{d}{dx}[(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4) - \cos x] = \sin x - (x - \tfrac{1}{6}x^3),$$

and so on.

Now, in (1), $1 - \cos x$ is positive and consequently $x - \sin x$ increases with x; it is a continuous function and vanishes for $x = 0$; thus $x - \sin x$ is positive for positive values of x. Apply this to (ii); and by a similar argument $\cos x - (1 - \frac{1}{2}x^2)$ is positive for positive values of x; and this combined with (iii) shows that $\sin x - (x - \frac{1}{6}x^3)$ is positive: passing to (iv), $(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4) - \cos x$ is positive; and so on. Thus we have established that $\sin x$ lies between x and $(x - \frac{1}{6}x^3)$; $\cos x$ between $1 - \frac{1}{2}x^2$ and $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Clearly the number of terms may be increased to any extent; and we may either introduce the infinite series at once, or leave the results in the form of bounds within which $\sin x$ and $\cos x$ are contained.*

*For the purpose of numerical calculation the second form is often more useful than the series.

A similar process can be used in connection with arc $\tan x$; for we have

$$\frac{d}{dx}[(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5)-\text{arc}\tan x)]=1-x^2+x^4-\frac{1}{1+x^2}=\frac{x^6}{1+x^2},$$

$$\frac{d}{dx}[\text{arc}\tan x-(x-\tfrac{1}{3}x^3)]=\frac{1}{1+x^2}-(1-x^2)=\frac{x^4}{1+x^2},$$

and so, since both of these differential coefficients are positive, we can prove that* arc $\tan x$ lies between $(x-\frac{1}{3}x^3)$ and $(x-\frac{1}{3}x^3+\frac{1}{5}x^5)$.

By an exactly similar method we can find bounds for e^{-x} and $\log(1+x)$, x being positive; but there is here a logical difficulty, since the differentiation of a^x (and therefore of $\log x$) depends on the limit

$$\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n.$$

The evaluation of this limit by any accurate process leads to difficulties of much the same character as those encountered in the limits for $\cos x$ and $\sin x$. I have generally found it most convincing (in a first course) to approximate to the limit by numerical calculation, using 7– or 8– figure tables. Thus we get the approximate values

$$(1+\tfrac{1}{10})^{10}=2\cdot593,\ (1+\tfrac{1}{100})^{100}=2\cdot705,\ (1+\tfrac{1}{1000})^{1000}=2\cdot717,$$

$$(1-\tfrac{1}{10})^{10}=2\cdot868,\ (1-\tfrac{1}{100})^{100}=2\cdot732,\ (1-\tfrac{1}{1000})^{1000}=2\cdot719,$$

which indicate† that the limit is in the neighbourhood of 2·718. Taking the limit as known, the differentiation of a^x, $\log x$, etc., is easy by the methods explained in any good text-book on the Calculus (such as Professor Gibson's).

It may be not out of place to record my opinion that, to a beginner, numerical verification of general theorems on series is often more convincing than the algebraic proofs. As a matter of personal experience, I found my own faith greatly strengthened by verifying that, to three places of decimals, the sine-series gave 1·000 and the cosine-series 0·000 for $x=1\cdot571$; but, as a practical matter, for economy of time smaller values of x will be found preferable, such as $\sin(\cdot5236)=\cdot500$, $\cos(\cdot5236)=\cdot866$, etc.

The sequence of propositions sketched here would, I think, be found to fit in very well with the *Report of the M.A. Committee on Higher School Mathematics* (*Gazette*, vol. 3, July, 1904, p. 52); and it is very likely that the main ideas have already been used by some teachers.‡ The method has the double advantage of giving some elegant illustrations of the elementary rules of the Calculus and, at the same time, of lightening the work in Trigonometry.

In conclusion, I venture to express the opinion that, as a general rule, any work which involves double limits or uniform convergence should be omitted from the school course; or practically speaking, that "ϵ-proofs" should be left out. Such subjects appear to be altogether better suited to a later stage in mathematical training: and the more so because they are entirely unnecessary to anyone but professional mathematicians.§ If this suggestion is accepted, as a matter of course the infinite products for $\sin x$ and $\cos x$ would

*The angle is taken (as usual in the Calculus) to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

†They also serve to show that the convergence to the limit is very slow, compared with that of the exponential series.

‡Some of the results (perhaps all of them) are given as exercises in Prof. Gibson's excellent *Calculus*; I had, however, obtained them independently before his book was published.

§It is perhaps going too far at present to suggest that the theory of double limits and uniform convergence is essential for a professed mathematician in England; just as, for the present, it seems hopeless to expect even an elementary knowledge of the foundations of geometry from our "up-to-date" Euclids.

be omitted (compare the last paragraph of the *Report* quoted above). But, so long as such proofs must be taught, I would strongly recommend the arrangement given in J. Tannery's admirable *Introduction à la Théorié des Fonctions d'une Variable*, a text-book which deserves to be better known in this country than it appears to be.*

T. J. I'a BROMWICH.

MATHEMATICAL NOTES.

153. [P. 3. b.] *On a Fundamental Theorem in Inversion.*

If two figures F, F' be inverses of each other, their inverses with respect to any pair of corresponding points P, P' are similar.

Demonstration. Let O be the centre, and ρ the radius of inversion of F, F'. Let Q, R be any two points of the figure F; and Q', R' the corresponding points of the figure F'. Also let q, r be the inverses of Q, R with respect to centre P, radius of inversion λ; and q', r' the inverses of Q', R' with respect to centre P', radius of inversion λ'. The theorem will be proved if we show that the ratio of qr to $q'r'$ is constant.

Use the following lemma: If A', B' be the inverses of A, B with respect to a centre O, radius of inversion ρ, then $A'B'/AB=\rho^2/OA\,.\,OB$.

We have by the lemma

$$\frac{qr}{QR}=\frac{\lambda^2}{PQ\,.\,PR}; \quad\text{and}\quad \frac{q'r'}{Q'R'}=\frac{\lambda'^2}{P'Q'\,.\,P'R'};$$

$$\begin{aligned}\therefore\ \frac{qr}{q'r'}&=\frac{\lambda^2}{\lambda'^2}\cdot\frac{QR}{Q'R'}\cdot\frac{P'Q'}{PQ}\cdot\frac{P'R'}{PR}\\ &=\frac{\lambda^2}{\lambda'^2}\cdot\frac{\rho^2}{OQ'\,.\,OR'}\cdot\frac{OP'\,.\,OQ'}{\rho^2}\cdot\frac{OP'\,.\,OR'}{\rho^2}\\ &=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'^2}{\rho^2}=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'}{OP}\ \text{since } \rho^2=OP\,.\,OP'\\ &=\frac{\lambda^2}{OP}\div\frac{\lambda'^2}{OP'}.\end{aligned}$$

Thus the ratio of qr to $q'r'$ *is* constant and therefore the theorem is proved.

A second demonstration will be given later on. The metrical relation obtained above, namely, that, if λ, λ' be the radii of inversion at P, P' the corresponding lengths in the inverses of F, F' with respect to P, P' are to one another as $\dfrac{\lambda^2}{OP}=\dfrac{\lambda'^2}{OP'}$ should be noted, as it is of importance in the applications of the theorem to recent Geometry.

From the theorem it follows that if a figure F undergoes n successive inversions and is thereby converted into F_n, and to the point P in F corresponds the point P_n in F_n, then the inverse of F_n with respect to P_n is similar to the inverse of F with respect to P.

154. [P. 3. b.]. *Continued Inversion by Coaxal Circles.*

An interesting way of obtaining some of Mr. C. E. Youngman's results (p. 7) is by the stereographic projection. Let HK be the points common to the coaxal system. Bisect HK at O, and draw OV equal to OH and perpendicular to the plane of the system. Draw a sphere through V with O as

*It may not be out of place to refer to the fact that the real difficulty in finding the infinite product is not to prove that $x(1-x^2/\pi^2)$, etc. are factors; but to prove that *no factor of the form e^{ax} is present.* This remark is due to Stolz, but is not made in any of the English text-books.

centre. To find the position of P after successive inversions with respect to the circles (A), (B), (C),... we join VP cutting the sphere in Q; bring Q to the position of Q' by successive reflections in the planes through HK perpendicular to VA, VB, VC,...; and then join VQ' cutting the plane AHK in the required position P'. Since successive reflexions in two planes through HK inclined at an angle a are equivalent to a rotation through $2a$ about HK, and since the circles (A) and (B) cut at an angle AVB = the angle between the planes through HK perpendicular to VA and VB, we have at once $[AB]=[CD]$, provided the circles (A) and (B) cut at the same angle as (C) and (D). If (C) or (D) is the line HK we have Mr. R. F. Davis's result. Again, if (A) and (B) are orthogonal $[AB]=[BA]$, for rotations through π and through $-\pi$ about HK are equivalent, etc.

HAROLD HILTON.

155. [K. 20. a.]. *Definitions of Trigonometrical Ratios, and General Proof of Addition-Theorems for Sine and Cosine.*

The object of this note is to indicate a method of treating the fundamental theory of Trigonometry which, so far as simplicity and completeness are concerned, seems to have considerable merit. No absolute novelty can be claimed for or alleged against it. (See, *e.g.*, Casey's *Trigonometry*.)

We shall use the symbol $\equiv$ as denoting "has the same length, direction and sense as"; and the symbol $=$ will as usual denote both equality in magnitude and likeness in sign.

Prop. I. The projections on a *directed* line Ox of any two directed line-segments of equal length and the same direction, are equal to one another.

Proof. Let AB, CD be two such segments, and ab, cd their projections on Ox.

Let AE and CF drawn $\parallel$ to Ox meet Bb and Dd in E and F. Then the triangles AEB, CFD are equal in all respects, having their corresponding sides in the same directions.

Thus $$AE \equiv CF; \quad \therefore\ ab \equiv cd.$$

Prop. II. If a directed line meet the directed line Ox in A, and any two segments of the former, starting from A, be taken, then the ratio of projection on Ox to segment is the same for the two segments.

Proof. Let AB, AG be two such segments (which may have like or unlike senses) and Ab, Ag their projections on Ox.

Then $$\frac{gA}{Ab}=\frac{GA}{AB}; \quad \therefore\ \frac{Ag}{AG}=\frac{Ab}{AB}.$$

Prop III. The ratio of projection on Ox to segment is the same in magnitude and sign for any two segments of any two directed lines that make the same angle a with Ox.

Proof. If LM and PQ be such segments, and the line LM meet Ox in A; and if along this line we lay off segments $AN \equiv LM$ and $AR \equiv PQ$, then the ratio of projection to segment is the same for LM and for AN (Prop. I.) and for AR (Prop. II.) and for PQ (Prop. I.).

Def. I. The ratio of projection on Ox to segment, for any directed segment of a directed line making an angle a with the directed line Ox, is the *cosine of* a.

Note that a directed segment of a directed line is positive or negative according as its sense is the same as that of the line or opposite to it. We have shown in Props. I., II., and III. that this definition of $\cos a$ is complete.

Def. II. The corresponding ratio when the projection is on a line making an angle $+90°$ with Ox is called $\sin a$.

Prop. IV. $\text{Cos}(\alpha \pm 180^\circ) = -\cos\alpha$.

For $\cos(\alpha + 180^\circ)$ is the cosine of the angle that a line inclined at α to Ox makes with the line inclined at -180° to Ox, *i.e.* with xO; and the projection of any segment on Ox is $= -$(its projection on xO).

Prop. V. $\text{Cos}(-\alpha) = \cos\alpha$.

The proof is obvious, if we take segments of equal magnitude starting from the same point of Ox.

Cor. 1. $\text{Sin}(-\alpha) = \cos(-\alpha - 90^\circ) = \cos(\alpha + 90^\circ) = \cos(\alpha - 90^\circ + 180^\circ)$
$$= -\cos(\alpha - 90^\circ)$$
$$= -\sin\alpha.$$

Cor. 2. $\text{Sin}(\alpha - 180^\circ) = -\sin(-\alpha + 180^\circ) = -\cos(-\alpha + 90^\circ) = -\cos(\alpha - 90^\circ)$
$$= -\sin\alpha.$$

Cor. 3. $\text{Cos}(\alpha + 90^\circ) = \cos(-\alpha - 90^\circ) = \sin(-\alpha) = -\sin\alpha$.

Prop. VI. $\text{Cos}(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Let
$$X\hat{O}H = \alpha, \quad H\hat{O}P = \beta, \quad H\hat{O}K = 90^\circ;$$
$$\therefore\ X\hat{O}P = \alpha + \beta, \quad X\hat{O}K = \alpha + 90^\circ.$$

Let N and M be the projections of P on OH and OK.

Now projection on Ox of OP = projection of ON + projection of NP

= ,, ON + ,, OM

$= ON\cos\alpha + OM\cos(\alpha + 90^\circ)$ (by Def. I.)

$= ON\cos\alpha - OM\sin\alpha$ (by Prop. V., Cor. 3).

Again, $ON = OP\cos\beta$; and $OM = OP\cos(\beta - 90^\circ) = OP\sin\beta$.

Hence projection on Ox of $OP = OP\cos\alpha\cos\beta - OP\sin\alpha\sin\beta$.

$\therefore$ by Def. I., $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Cor. 1. $\text{Cos}(\alpha - \beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Cor. 2. $\text{Sin}(\alpha - \beta) = \cos(\alpha - \beta - 90^\circ)$
$$= \cos\alpha\cos(\beta + 90^\circ) + \sin\alpha\sin(\beta + 90^\circ)$$
$$= -\cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Cor. 3. $\text{Sin}(\alpha + \beta) = \sin\{\alpha - (-\beta)\} = \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta)$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Definitions III., IV., V., VI. :—

Tan α is the ratio $\sin\alpha : \cos\alpha$,

cot α ,, $\cos\alpha : \sin\alpha$,

sec α ,, $1 : \cos\alpha$,

cosec α ,, $1 : \sin\alpha$.

R. F. Muirhead.

REVIEWS.

Hermann Grassmann. **Gesammelte mathematische und physicalische Werke.** II. Band, I. Theil. **Die Abhandlungen zur Geometrie und Analysis.** 1904. (Teubner, Leipzig.)

Hermann Grassmann was born in Stettin in 1809, his father being the mathematical professor at the Gymnasium, and the author of several text-books. On his father's death the son succeeded to his position, and remained in Stettin till his death in 1877.

It may come as a surprise to some that the Stettin schoolmaster's scientific writings are so extensive as to be republished in six substantial volumes. During his lifetime and for some years after his death little attention was paid to his works, partly owing to their novelty, partly to the obscure style in which they were written. In the progress of mathematical thought the same ideas have been reached by other channels, so that Grassmann's writings have become much more intelligible than they were to his contemporaries, and we can marvel at the prophet who was so much in advance of his time.

The edition of Grassmann's works which is now in course of publication owes its inception to Klein, who, in 1892, persuaded the Royal Philosophical Society of Saxony to authorise the undertaking, and secured the services of Engel as general editor, assisted by Lüroth, Study, Scheffers and two sons of the author.

It will represent the English usage more closely if we translate "Band" by "Part" and "Theil" by "Volume." The first volume contains the "Ausdehnungslehre" of 1844 and the second that of 1862; papers on Geometry and Analysis are collected in the volume under review, and papers on Mechanics and Physics in the fourth. The third part and last two volumes, which have not appeared yet, will contain the Prüfungsarbeit on Tides, and matter of a critical and historical character.

Nearly three hundred pages are occupied by papers of a geometrical or algebraical nature, chiefly applications of the methods of the Ausdehnungslehre to the theory of plane curves, in particular, to their linear generation. The next fifty pages contain extracts from a school text-book on Arithmetic, published in 1861, followed by a short extract from one on Trigonometry published in 1865. A list is given of the most important alterations that have been made from the original editions, and this is followed by ample notes of an explanatory nature.

R. W. H. T. Hudson.

G. Holzmüller. **Vorbereitende Einführung in die Raumlehre.** Pp. 123. 1904. (Teubner, Leipzig.)

It is explained on the title page that this little book is published "Im Anschluss an die preussischen Lehrpläne von 1901 zur freien Auswahl für den Anfangsunterricht bearbeitet und mit Anleitungen zum Herstellen von Unterrichts-Modellen versehen." Everyone agrees nowadays that a course of "observational" or "practical" geometry should precede the logical theory; but there is a considerable danger of the reasoning faculty being altogether sacrificed to the cultivation of geometrical intuition. The two are quite distinct and even opposed to each other, and the fact that they are customarily exercised upon the same subject matter introduces great difficulties, although it is a necessary means of supplying sufficient interest.

This book begins with a careful discussion of the elements of space, the ultimate appeal being to intuition. The greater part of the book is devoted to constructions in two and three dimensions. The author rightly observes that the difficulties of "solid" geometry are usually over-estimated. It is only after long years of artificial restriction to the abstract plane that the power of realising space configurations is enfeebled: it should be encouraged while it is still natural.

R. W. H. T. Hudson.

Leçons élémentaires sur la Théorie des Fonctions Analytiques. Edward A. Fouët. Seconde Partie. Pp. 299. 1904. (Gauthier-Villars.)

In this, his second, volume M. Fouët includes the last two chapters of the first book and the first three chapters of Book II. The first chapter contains a general discussion of functions defined by means of differential equations, and includes a short account of the work of MM. Fuchs, Poincaré and Painlevé, in connection with the existence of parametric critical points (pp. 34 *sqq*). M. Painlevé demonstrated that out of equations of the form $y''=R(y', y, x)$ where R is rational in y', algebraic in y, and analytic in x, he has determined explicitly all those of which the essential critical points are fixed, and that where, in such cases, the general integral is an essentially new uniform function, the equations can be reduced to three simple types. He suggested also the way of discussing equations of higher order.

Painlevé's work in connection with his three types of differential equations has appeared within the last few years (1900-1903), so that there would seem to be

furnished a comparatively new field in the discussion of the properties of the new transcendants he has thus defined. M. Fouët, however, fails to give any account of Painlevé's methods, and ignores the work of Picard and of Mittag Leffler dealing with the same subject (*C. R.* vols. CII., CIV., etc. *Acta Math.* vols. XVII., XVIII., etc.). In this first chapter M. Fouët gives certain theorems on the existence of integrals of partial equations, and concludes with theorems connected with Dirichlet's Problem.

The last chapter of Book I. is concerned with functions defined by functional properties, such as, for example, properties of periodicity. Two pages are devoted to the consideration of doubly periodic functions.

In the three chapters of Book II. given in this volume, analytic functions are discussed from the points of view of Cauchy, Weierstrass and Riemann.

M. Fouët's main difficulty has apparently arisen in connection with the arrangement of the work. He is perhaps justified in his attempt to give separate discussions of analytic functions from the three different standpoints, and this attempt necessitates a general account of function theoretic ideas on the lines of that given in Book I. But the effect of his arrangement is a little unfortunate in such cases as the following:—

The discussion of the function $\Gamma(z)$ is divided into two parts, one of which appears in Book I. and the other in Book II. The elementary discussion of infinite products, which is essentially connected with Weierstrass' point of view of function theory, is separated from it by such matter as a general discussion of differential equations, and an account of Dirichlet's problem. This problem itself, which is intimately connected with Riemann's, is not treated in connection with the account of that work, but in connection with differential equations; and an account of the properties of harmonic functions, which should be given in connection with the discussion of Laplace's Equation, is reserved to make an addition to the chapter on Riemann's theory of functions. If M. Fouët intends to discuss in detail such an important branch of his subject as elliptic functions, it would be an advantage to have the account of the whole subject in one part of the treatise. Instead of this he has considered functions possessing an algebraic addition theorem in Vol. I., p. 168, and in Vol. II., pp. 83, 84, while the Weierstrassian σ function is given as a product in Vol. II., p. 188, general discussion being apparently reserved for a later volume.

The work is chiefly interesting for its account of many of the outlying parts of the subject, such as, for example, that of minima surfaces (Vol. II., pp. 285 *sqq.*). It is unfortunate that, owing to the scope of the work, such accounts are necessarily brief.

J. E. WRIGHT.

Leçons sur les séries à termes positifs. Par. É. BOREL. Pp. 91. 1902. (Paris, Gauthier-Villars.)

This is a reproduction of 20 lectures given by M. Borel at the *Collège de France* in the session 1900-1901; they have been collected and edited by one of M. Borel's class, M. Robert d'Adhémar. It forms a part of the series of works entitled *Nouvelles leçons sur la théorie des fonctions*, in which have appeared M. Borel's lectures on integral functions and on divergent series.

In chapter 1 we are concerned mainly with the familiar logarithmic criteria of convergence, which M. Borel attributes to Bertrand, though they appear to have been given first by de Morgan (see Chrystal's *Algebra*, vol. 2, historical note to chap. xxvi.). It is not until we reach the middle of chapter 2 that M. Borel introduces us to the idea which forms the central thread of the book; this idea is that of the *croissance* (degree of rapidity of increase) of a function always increases with the independent variable x. For the simple power x^n, the *croissance* is represented by the index n; for the exponential e^x, a symbol ω is introduced, which must [in virtue of the property $\lim_{x=\infty}(x^n/e^x)=0$] be greater than any ordinary index; the symbol ω has also been used by G. Cantor to represent the *transfinite number*, which gives a useful analogy for the exponential *croissance*. The laws of combination of the symbol ω are investigated, and the results are tabulated on p. 47; they agree only partially with the algebraic laws of ordinary numbers.*

* *E.g. Multiplication* is defined as in the following examples: ωn is the *croissance* of exp. (x^n), but $n\omega$ is that of $(\text{exp. } x)^n$; ω^2 is that of exp. (exp. x); and so on. Thus $\omega n \neq n\omega$; but $\omega^\alpha\omega^\beta = \omega^{\alpha+\beta}$.

Passing to chapter 4 we find two simple criteria for the convergence of double series (with positive terms) which may be quoted here; taking the series $\Sigma\Sigma v_{mn}$ ($m, n=1, 2, \ldots \infty$) we have a *sufficient* test of convergence if, after a certain stage, we can say that $v_{mn} < (m+n)^{-(2+p)}$, where $p>0$; or that $v_{mn} < \Sigma 1/(m^{2+p}+n^{2+p})$. But if $v_{mn} > (m+n)^{-2}$, the series diverges. Similar conditions are then obtained for double integrals, in which the area of integration extends to infinity.

In chapter 5 we take up the question of the *croissance* of a power-series $\Sigma a_n x^n$; it is shown that if $\phi(n) = a_n^{-1/n}$ is an increasing function, of order n^p, then the *croissance* of the function defined by the series is $\omega(1/p)$; but the inverse problem is much harder.* Another interesting result is given for the case in which $\phi(n)$ tends to the limit 1 as n tends towards ∞; taking the series Σa_n to be divergent, the behaviour of the function $\Sigma a_n x^n$ near $x=1$ is determined. For instance,

$$\lim_{x=1}[(1-x)^p(1^{p-1}x+2^{p-1}x^2+\ldots+n^{p-1}x^n+\ldots)]=\Gamma(p).$$

The remainder of the chapter can hardly be appreciated without a direct reference to the original; the book terminates with a short account of analogous properties of the double series $\Sigma a_{mn}x^m y^n$.

We have no hesitation in recommending those who are interested in modern analysis to add this book to their shelves; it is a worthy successor to M. Borel's earlier text-books; a higher recommendation cannot be given.

T. J. I'a BROMWICH.

Théorie élémentaire des séries. Par M. GODEFROY. Pp. 266. 1903. (Paris, Gauthier-Villars.)

This book differs entirely in its aims from M. Borel's; as appears both from the title and the method of treatment, it is mainly intended for beginners in the field of analysis. It is somewhat similar to the treatment given in Chrystal's *Algebra* and Hobson's *Trigonometry*, except that the complex variable is nowhere used;† in this respect following the example of the first edition of Stolz's *Allgemeine Arithmetik*. However, M. Godefroy assumes a knowledge of the elements of the Differential Calculus, and so is able to give some theorems which are not mentioned in the treatises named; for example (p. 75) we have the theorem that a linear differential equation is soluble by power-series which converge within the same interval as the power-series which represent the co-efficients. Some of the simpler properties of Bessel's and Legendre's functions (of the first kind) are obtained.

It is something of a novelty to find in a professedly elementary book, a very clear and concise discussion of the Weierstrassian continuous, non-differentiable, function

$$\Sigma r^n \cos(a^n \pi x),$$

where $0<r<1$ and a is an an odd integer such that $ar>1+\frac{3}{2}\pi$. According to a footnote to this article, M. Poincaré has remarked that a hundred years ago a function such as this would have been thought an outrage on common sense; but we fear that in England a much more recent date might be assigned.

M. Godefroy concludes his book with 50 pages on the gamma-function; for the most part, these articles appear to be reprints of those in his earlier work bearing that title (see *Math. Gazette*, March, 1902).

As a whole the work should be found extremely useful by teachers who have to lecture on the elementary theory of infinite series; and I have made considerable use of it in my course of lectures on the subject. T. J. I'a BROMWICH.

Theorie und Praxis der Reihen. By C. RUNGE. Pp. 266. 1904. (Goschen, Leipzig.)

In this admirable monograph Professor Runge has devoted his attention to the practical use of series at the expense of the general theory and all its details. The following is a summary of the five chapters into which the book is divided.

* For the function $\phi(n)$ may be changed so as to increase quite irregularly without in any way affecting the *croissance* of the series.

† This will be regarded as an advantage by those teachers who think it well to begin the study of power-series before introducing the complex variable.

Chap. I. Series with constant terms. Convergence. Operations on series with extension to series with complex terms. Chap. II. Uniform convergence (with graphical illustrations). Power series. Recurrence. Reversion. Integration and Differentiation of uniform series. Cauchy's theorem on the integration of a uniform function of a complex variable with applications. Cylindrical functions of one variable. Interpolation series. Chap. III. A simple treatment of Fourier's series. Michelson and Stratton's Harmonic Analyser. Chap. IV. Infinite Products. Circular and theta functions. Chap. V. Development in series of functions of more than two variables. Extension of Taylor's series and others. Spherical functions.

A useful and interesting little volume.

CORRESPONDENCE.

To the Editor of the "Mathematical Gazette."

Civil Service Commission,
Burlington Gardens, W., 23/9/04.

Dear Sir,—Once there was no law in nature, and every thing did what seemed good in his own eyes. But some things said, "This is not good. Behold the civilization of man. Let us also have laws." And a conference was held, and to it came representatives of all forms of matter and energy. And at the conference they said, "There is a good and just man called Isaac Newton. He would make just laws and not severe." And they asked him. And he made laws just and not severe, and wrote them in the Latin tongue. And the conference accepted the laws and said all things must obey, even things mental and spiritual, for (they said) why did they not come to the conference? So all nature obeyed.

In course of time it happened that none knew the Latin tongue, and none knew the law to obey it. So the two chief priests read Latin diligently, and put the laws into the common tongue, and wrote commentaries on them so precise that no thing dared say he did not understand. And there was peace.

Such a myth would explain our recent teaching of mechanics. The Committee of the Mathematical Association deserve our thanks for rebelling and founding the science again on experiment. But here and there their view might be made clearer. Suggestion 22 might be thought to support the old method. And it seems a pity, after obtaining experimental results in Part A, to discard in Part C all but the parallelogram of forces and to deduce all others from this in a way that would gladden the heart of Euclid.—Yours faithfully

David Mair.

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